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Prolongation Theory. A New Nonlinear Schrödinger Equation

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We discuss a new kind of nonlinear Schrödinger equation from the viewpoint of prolongation theory. It is shown that the equation possess a Lax pair with a 3×3 matrix structure. It is further demonstrated that by a multiple scale perturbation of Zakharov *et al.* it can be reduced to the usual KdV equation.

1. INTRODUCTION

In the continuing attempt to discover new integrable systems, people have taken recourse to various methodologies. Of late a new kind of nonlinear Schrödinger equation has been discussed by Hiroto (19) by his bilinearization approach. But it has also been demonstrated that many equations for which Hiroto's method works and N-soliton solutions can be obtained are not completely integrable in the sense of Painlevé or they do not have a Lax pair. One of the most important equations is that of Zakharov for ion, acoustic solution (Goldstein and Infeld, 1984; Roy Chowdhury and Roy, 1979). For this system both the above statements are true. Here we make a prolongation analysis for the new nonlinear Schrödinger equation whose N-soliton solution was obtained by Hiroto and prove that an inverse scattering problem exists for this system with a 3×3 matrix structure. Finally we try to understand the connection of this new system with an earlier integrable case following the approach of Zakharov and Kuznetsov (1986) and observe that our equation is connected to the simple KdV system via multiple scale perturbation.

The equation under consideration reads

$$-i\phi_t + \phi_{xx} + \frac{2|\phi_x|^2\phi}{1 - \phi\phi^2} = 0 \tag{1}$$

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Before proceeding to the detailed analysis, let us note that an equation with the same type of nonlinearity but which is Lorentz invariant in two dimensions was initially solved by Getamanov, and later such equations were discussed in the context of a generalized Lund–Regge system, but none of them had this linear dependence on the time derivative or "nonrelativistic" structure. Let us consider equation (1) along with its complex conjugate

$$i\phi_t^* + \phi_{xx}^* + \frac{2|\phi_x|^2\phi^*}{1-\phi\phi^*} = 0 \tag{2}$$

We then define the basis-independent set of variables ($\phi_x = z$, $\phi_x^* = z^*$). Then equations (1) and (2) are equivalent to the following set of differential forms:

$$\begin{aligned} \alpha_1 &= d\phi \wedge dt - z\, dx \wedge dt \\ \alpha_2 &= d\phi^* \wedge dt - z^*\, dx \wedge dt \\ \alpha_5 &= i\, d\phi \wedge dx - dz \wedge dt + \frac{2zz^*\phi}{1-\phi\phi^*}\, dx \wedge dt \\ \alpha_6 &= -i\, d\phi^* \wedge dx + dz^* \wedge dt + \frac{2zz^*\phi^*}{1-\phi\phi^*}\, dx \wedge dt \end{aligned} \tag{3}$$

If we now proceed to search for one-form ω,

$$\begin{gathered} \omega = dy + F\, dx + G\, dt \\ F = F(z, z^*, \phi, \phi^*, y), \qquad G = G(z, z^*, \phi, \phi^*, y) \end{gathered} \tag{4}$$

such that

$$d\omega = \sum f_i \alpha_i + (\bar{a}\, dx + \bar{b}\, dt) \wedge \omega \tag{5}$$

then it can be very easily observed that there is no nontrivial solution for F and G. Such a situation may be seen in other equations, but we have not found any other example of this sort. So we now proceed by considering the derived equations for (1) and (2), which are

$$\begin{aligned} -i\phi_{tx} + \phi_{xxx} &+ \frac{2\phi_{xx}\phi_x^2}{1-\phi\phi^*}\,\phi + \frac{2\phi_x\phi_{xx}^*}{1-\phi\phi^*}\,\phi \\ &+ \frac{2|\phi_x|^2\phi_{xx}}{1-\phi\phi^*} + \frac{2|\phi_x|^2}{(1-\phi\phi^*)^2}(\phi_x\phi^* + \phi\phi_x^*)\phi = 0 \end{aligned} \tag{6}$$

and its complex conjugate.

To convert these into differential forms, we define further new basis variables

$$\phi_{xx} = z_x = p, \qquad \phi^*_{xx} = z^*_x = p^* \tag{7}$$

so that the equivalent set of forms are those given previously and the following:

$$\begin{aligned}
\alpha_3 &= dz \wedge dt - p\, dx \wedge dt \\
\alpha_4 &= dz^* \wedge dt - p^*\, dx \wedge dt \\
\alpha_7 &= i\, dz \wedge dx + dp \wedge dt + \frac{2z^*\phi p}{1-\phi\phi^*}\, dx \wedge dt \\
&\quad + \frac{2zp^*\phi}{1-\phi\phi^*}\, dx \wedge dt + \frac{2z^2z^*}{1-\phi\phi^*}\, dx \wedge dt \\
&\quad + \frac{2z^2z^*\phi^*\phi}{(1-\phi\phi^*)^2}\, dx \wedge dt + \frac{2zz^{*2}\phi^2}{(1-\phi\phi^*)^2}\, dx \wedge dt \\
\alpha_8 &= -i\, dz^* \wedge dx + dp^* \wedge dt + \frac{2z^*p\phi^*}{1-\phi\phi^*}\, dx \wedge dt \\
&\quad + \frac{2zp^*\phi}{1-\phi\phi^*}\, dx \wedge dt + \frac{2zz^{*2}}{1-\phi\phi^*}\, dx \wedge dt \\
&\quad + \frac{2z^2z\phi^{*2}}{(1-\phi\phi^*)^2}\, dx \wedge dt + \frac{2zz^*}{(1-\phi\phi^*)^2}\, dx \wedge dt
\end{aligned} \tag{8}$$

To obtain the prolongation structure we search for a one-form

$$\begin{aligned}
\omega = dy &+ F(\phi, \phi^*, z, z^*, p, p^*, x, t, y)\, dx \\
&+ G(\phi, \phi^*, z, z^*, p, p, x, t, y)\, dt
\end{aligned} \tag{9}$$

so that the exterior derivative $d\omega$ remains in the ideal generated by $\alpha_i(i = 1 \text{ to } 8)$, and

$$d\omega = \sum a_i\alpha_i + (\bar{a}\, dx + \bar{b}\, dt) \wedge \omega \tag{10}$$

Equating coefficients of basic differential forms, we obtain the following equations for the structure of F and G:

$$F_p = 0 = F_{p^*}, \qquad F_z = iG_p, \qquad F_{z^*} = -iG_{p^*} \tag{11}$$

If we now use the further assumption that F and G are not explicitly dependent on (x, t), then

$$G_0 z + G_{\phi^*}z^* + G_z p + G_{z^*}p^* + i(F_\phi p - F_{\phi^*}p^*)$$

$$+2iF_\phi \frac{zz^*\phi}{1-\phi\phi^*} - 2iF_{\phi^*} \frac{zz^*\phi^*}{1-\phi\phi^*} - 2iF_z \left\{ \frac{z^*\phi p}{1-\phi\phi^*} \right.$$

$$\left. + \frac{zp^*\phi}{1-\phi\phi^*} + \frac{z^2 z^*}{1-\phi\phi^*} + \frac{z^2 z^* \phi^* \phi}{(1-\phi\phi^*)^2} + \frac{zz^{*2}\phi^2}{(1-\phi\phi^*)^2} \right\}$$

$$+2iF_{z^*} \left\{ \frac{z^* p \phi^*}{1-\phi\phi^*} + \frac{zp^*\phi^*}{1-\phi\phi^*} + \frac{zz^{*2}}{1-\phi\phi^*} + \frac{z^2 z^* \phi^{*2}}{(1-\phi\phi^*)^2} \right.$$

$$\left. + \frac{zz^{*2}\phi\phi}{(1-\phi\phi^*)^2} \right\} + [F, G] = 0 \tag{12}$$

Also

$$F_{zz} = F_{z^*z^*} = 0, \qquad G_{z^*p} + G_{zp^*} = 0 \tag{13}$$

So we now set

$$F = zA + z^*B + D$$

$$G = zz^*M + PN + P^*R + zQ + z^*S + T \tag{14}$$

Now substituting these in equation (12) and equating various coefficients of z^2, zz^*, zp, etc., we get

$$M_\phi + \frac{2i\phi}{1-\phi\phi^*} A_\phi - \frac{2i\phi^2}{1-\phi\phi^2} A_{\phi^*} - \frac{2iA}{1-\phi\phi^*}$$

$$- \frac{2i\phi^*\phi A}{(1-\phi\phi^*)^2} + \frac{2i\phi^{*2}B}{(1-\phi\phi^*)^2} + [A, M] = 0 \tag{15}$$

along with

$$M_{\phi^*} + \frac{2i\phi}{1-\phi\phi^*} B_\phi - \frac{2i\phi^2}{1-\phi\phi^*} B_{\phi^*} - \frac{2i\phi^2 A}{(1-\phi\phi^*)^2}$$

$$+ \frac{2iB}{1-\phi\phi^*} + \frac{2i\phi^{*2}B}{(1-\phi\phi^*)^2} + [B, M] = 0$$

$$N_\phi + iA_\phi + [A, N] = 0$$

$$R_\phi - iB_{\phi^*} + [B, R] = 0$$

$$R_{\phi^*} + M - iA_{\phi^*} - \frac{2i\phi A}{1-\phi\phi^*} + \frac{2i\phi^* B}{1-\phi\phi^*} + [A, R] = 0$$

$$N_{\phi^*} + M + iB_\phi - \frac{2i\phi A}{1-\phi\phi^*} + \frac{2i\phi^2 B}{1-\phi\phi^2} + [B, N] = 0$$

$$S_\phi + Q_{\phi 2} + \frac{2i\phi}{1-\phi\phi^*} D_\phi - \frac{2i\phi^2}{1-\phi\phi^2} D_{\phi^2} + [A, S]$$
$$+[B, Q]+[D, M]=0 \tag{16}$$

An important solution for the functions A, B, D, etc., are found to be

$$\begin{aligned}
A &= \frac{\phi^*}{1-\phi\phi^*} x_1(y) + x_9(y) \\
B &= \frac{\phi}{1-\phi\phi^*} x_3(y) + x_{10}(y) \\
N &= \frac{i\phi^*}{1-\phi\phi^*} x_1(y) + ix_9(y) \\
R &= \frac{-i\phi}{1-\phi\phi^*} x_3(y) - ix_{10}(y) \\
D &= \frac{1}{1-\phi\phi^*} x_5(y) \\
Q &= \frac{1}{1-\phi\phi^*} x_6(y) \\
S &= \frac{1}{1-\phi\phi^*} x_7(y) \\
T &= \frac{1}{(1-\phi\phi^*)^2} x_8(y)
\end{aligned} \tag{17a}$$

whence M is given as

$$\begin{aligned}
M = {} & \frac{i}{1-\phi\phi^*}(x_1+x_2) + \frac{\phi\phi^*}{(1-\phi\phi^*)^2}\{3x_1 - x_3 + [x_1, x_3]\} \\
& + \frac{i\phi^*}{1-\phi\phi^*}\{2x_{10} + [x_1, x_{10}]\} + i[x_9, x_{10}] \\
& + \frac{i\phi}{1-\phi\phi^*}\{2ix_9 + [x_9, x_3]\}
\end{aligned} \tag{17b}$$

On substitution of these in equation (15) we are led to the following commutation rules for the incomplete Lie algebra:

$$[x_5, x_8] = [x_{10}, x_7] = [x_9, x_6] = 0$$
$$i[x_5, x_9] = x_7, \qquad x_5 + [x_5, x_1] = 0$$

$$
\begin{aligned}
i[x_5, x_{10}] &= -x_6, & x_5 + [x_5, x_3] &= 0 \\
[x_1, x_6] + x_6 &= 0, & x_7 + [x_3, x_7] &= 0 \\
[x_3, x_8] + 2x_8 &= 0, & [x_1, x_8] + 2x_8 &= 0 \\
[x_9, x_8] + [x_5, x_8] &= 0, & [x_{10}, x_8] + [x_5, x_7] &= 0
\end{aligned}
\tag{18}
$$

Closer of the Algebra

For obtaining the closed Lie algebra we invoke the scaling transformation symmetry of equation (1) (Shadwick, 1980). It is easily see that the equation remains invariant under $t' \to \lambda^2 t$, $x' \to \lambda x$ and $\phi' \to \phi$, whence for invariance of $\omega = dy + G\,dx + F\,dt$ we must have $F \to \lambda^{-1} F$, $G \to \lambda^{-1} G$, which immediately leads to the following isomorphism of the generators:

$$
\begin{aligned}
&x_1' \Rightarrow x_1, \quad && x_3' \Rightarrow x_9, \quad && x_9' \Rightarrow x_9, \quad && x_{10}' \Rightarrow x_{10} \\
&x_5' \Rightarrow \lambda x_5, && x_6' \Rightarrow \lambda x_6, && x_7' \Rightarrow \lambda x_7, && x_8' \Rightarrow \lambda^2 x_8
\end{aligned}
$$

Imposition of these properties on the commutation rules immediately suggests

$$
[x_5, x_8] = [x_6, x_8] = [x_7, x_8] = 0 \tag{19}
$$

Furthermore, the structure of the Jacobi identities suggests that we use

$$
x_1 = x_3, \qquad x_9 = x_{10}, \qquad x_6 = -x_7
$$

So finally we get

$$
\begin{aligned}
[x_5, x_9] &= i x_6, & [x_5, x_6] &= 0 \\
[x_1, x_5] &= x_5, & [x_5, x_8] &= 0 \\
[x_1, x_6] &= x_6, & [x_8, x_6] &= 0 \\
[x_1, x_8] &= -2x_8, & [x_8, x_9] &= 0 \\
[x_1, x_9] &= 0, & [x_9, x_6] &= 0
\end{aligned}
\tag{20}
$$

Further simplification can be obtained by setting $x_8 = 0$. Whence a 3×3 representation is obtained as

$$
\begin{aligned}
x_1 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad & x_5 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
x_6 &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & x_9 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}
\tag{21}
$$

So one can write down a matrix form of the Lax pair.

The Multiple Scale Transform

The equation studied above is usually studied with the linear part replaced by a Klein-Gordon operator $\partial_t^2 - \partial_x^2$, and such equations are known to be variants of nonlinear σ-models. But in our case due to the occurrence of the Schrödinger operator $i\partial_t + \partial_x^2$ in the linear part we here analyze a possible connection of such an equation is known integrable system via an approach of Zakharov *et al.* Let us set

$$\phi = \sqrt{N}\, e^{i\psi}$$

Then the real and imaginary parts of (1) are

$$N_t + 2(\partial/\partial x)(NV) = 0$$

where $V = \psi_x$ and

$$\psi_t + \frac{1}{4}N^{-2}N_x^2 - \tfrac{1}{2}N_{xx} + \psi_x^2 - \frac{2}{1-N}\left(\frac{1}{4}N^{-1}N_x^2 + N\psi_x^2\right) = 0 \qquad (22)$$

Now, according to Zakharov and Kuznetsov (1986), we make a scale transformation of the form

$$\begin{aligned}
x' &= \varepsilon(x-t) \\
t' &= \varepsilon^3 t \\
N &= 1 + \sum_{k=1}^{\infty} \epsilon^{2k} n_k(x', t') \\
V &= \sum_{k=1}^{\infty} \varepsilon^{2k} v_k(x', t') \\
\psi &= -t + \sum_{k=1}^{\alpha} \varepsilon^{2k-1} \psi_k(x', t')
\end{aligned} \qquad (23)$$

where ε is a small parameter; collecting powers of ε^5, we immediately obtain

$$n_{1t'} + n_1 n_{1x'} - \tfrac{1}{4} n_{1x'x'x'} = 0 \qquad (24)$$

with $2V_1 = -n_1$. Equation (24) is nothing but a form of KdV equation. So, as suggested by Zakharov *et al.*, if the transformation (23) is performed on an integrable system, then another integrable system is obtained. On the other hand, sometimes new integrable systems may come from nonintegrable systems through (23).

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